On the Number of Connected Components of Joint Degree Matrix Realizations

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Introduction

Joint Degree Matrix

- Joint Degree Matrix (JDM) for a graph $G(V, E)$:

$$JDM(k,l) = \sum_{u \in V_k} \sum_{v \in V_l} 1_{\{(u,v) \in E\}}$$

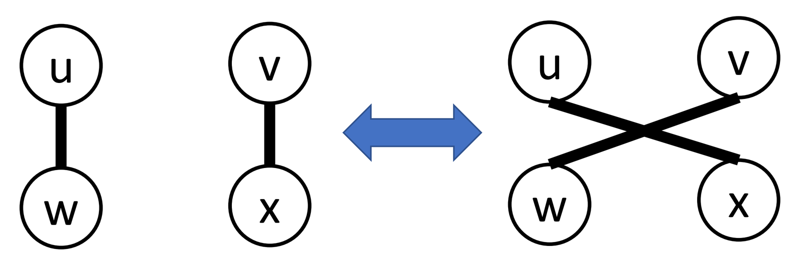
$$V_i \coloneqq \{v \in V | degree(v) = i\}$$

JDM Realizations

- Simple graphs with the same target Joint Degree Matrix (JDM) are called realizations of that JDM

JDM-preserving double-edge swap

- Given four distinct nodes (u, v, x, w) and (u,w), (v,x) edges [and (u,x), (v,w) are not edges]. Deleting (u,w), (v,x) and adding (u,x), (v,w) preserves JDM if $u, v \in V_i$

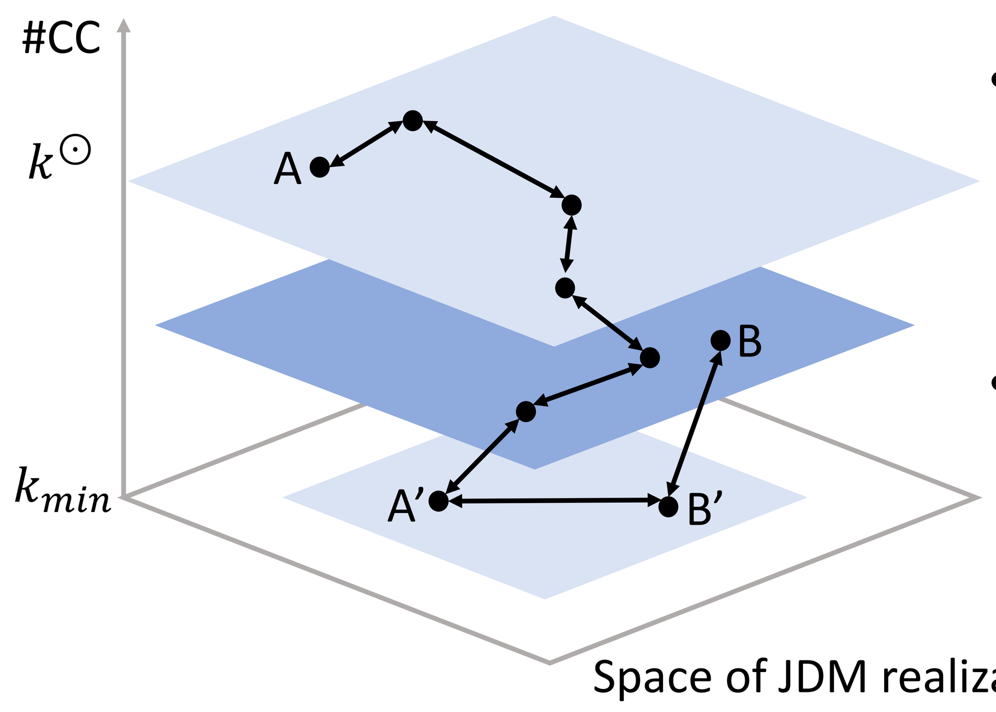


Prior Work

- Space of realizations is connected over JDM-preserving double-edge swap operation:
  - Symmetric difference based proof [Amanatidis '15]
  - Balanced realization based proof [Czabarka '15]
- Construct JDM realization with single connected component [Amanatidis '15, '18]

Main result

- ***Theorem.*** *The space of simple, undirected graphs with a target JDM and no more than $k^{\odot}$ number of connected components is connected under a sequence of JDM-preserving double-edge swaps.*



Space of JDM realizations

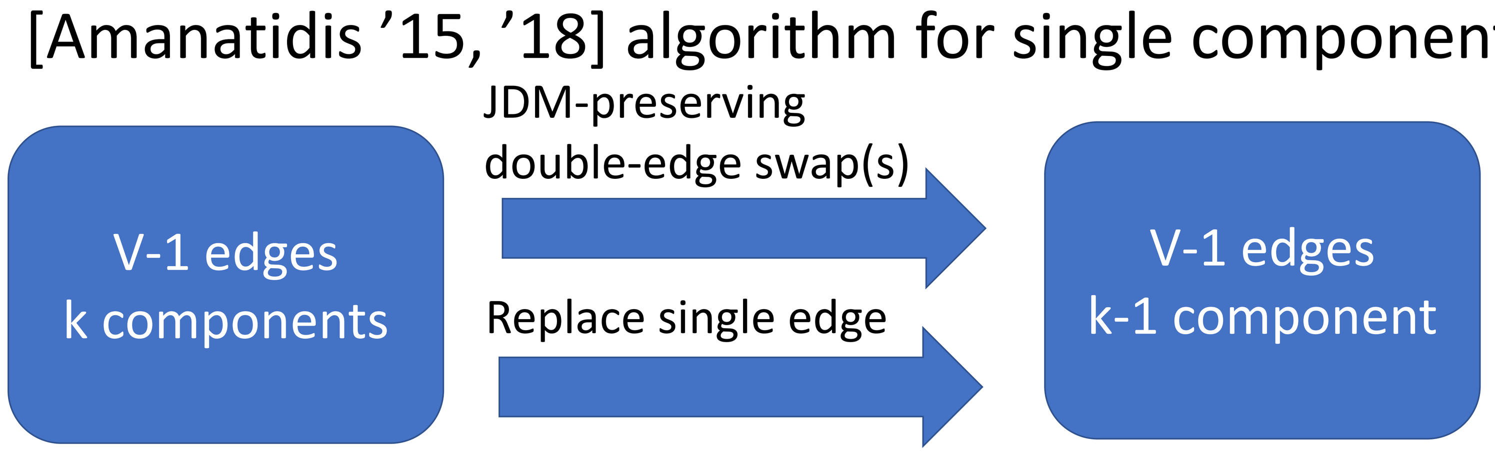
- Use Lemma 1 to transform JDM realizations A and B to A' and B' with minimum number of connected components
- Use Lemma 2 to transform A' to B' while preserving number of connected components.

- ***Lemma 1.*** *For a realizable JDM, there exists a JDM-preserving double-edge swap sequence that transforms any realization to a realization with minimum number of connected components, such that there is no double-edge swap that increases the number of connected components.*

- ***Lemma 2.*** *The space of JDM realizations with minimum number of connected components, k_{min}, is connected under JDM-preserving double-edge swaps.*

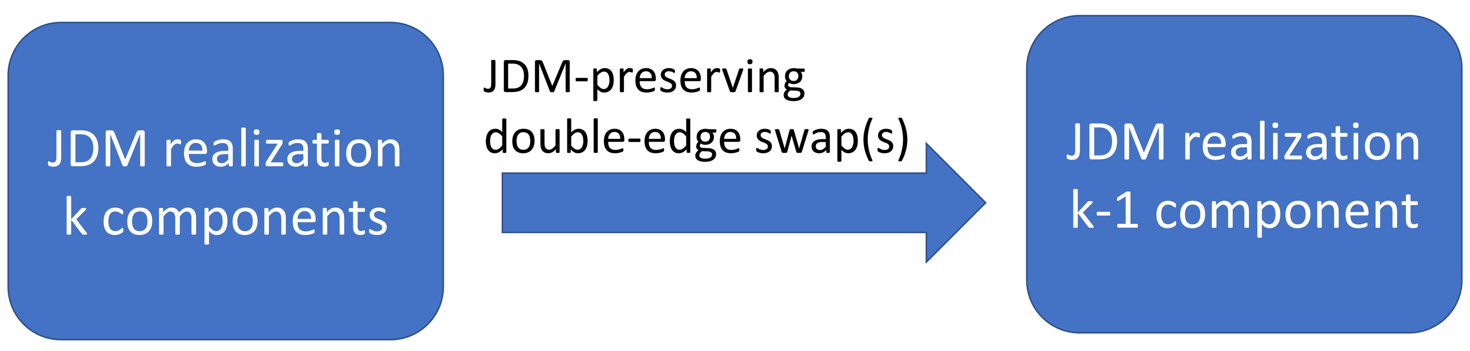
Proof Sketch of Lemma 1

- [Amanatidis '15, '18] algorithm for single component:

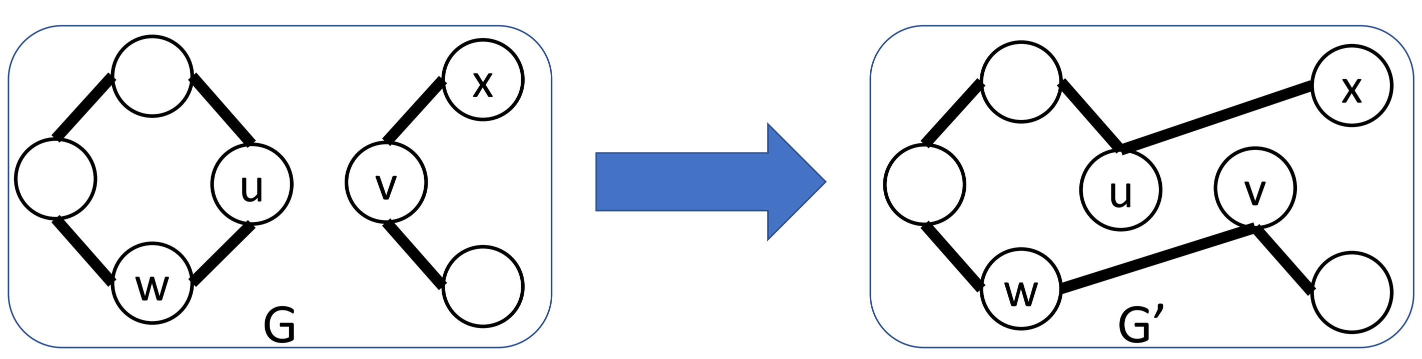


Repeat until either k=1, a spanning tree for target JDM or produce certificate that single connected component realization doesn't exist

- Our algorithm for minimum number of components:



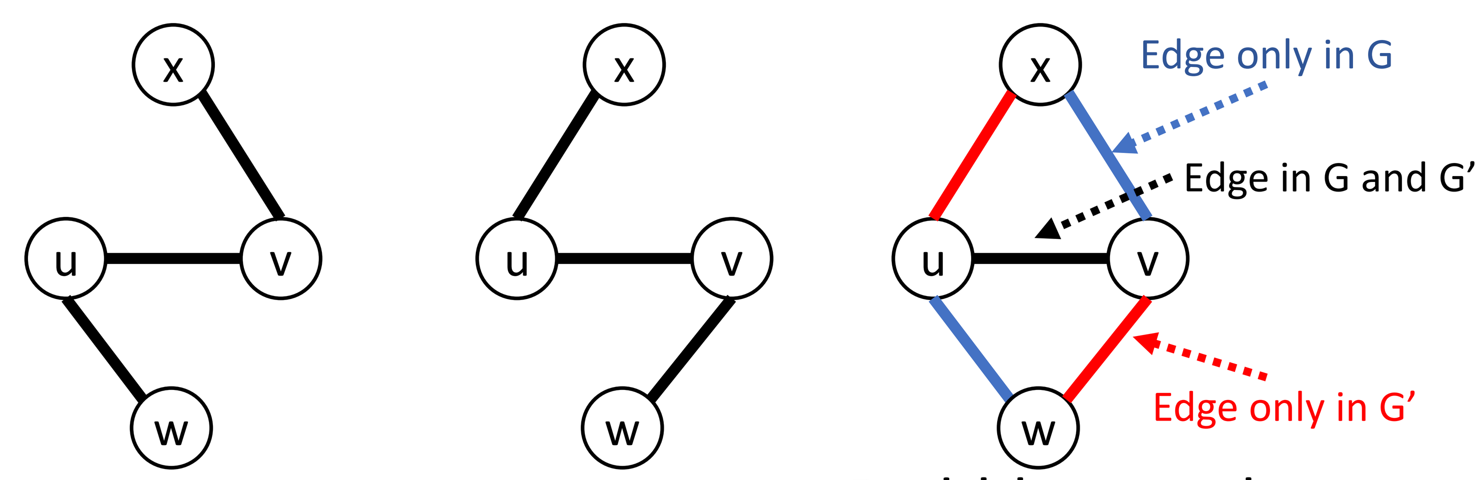
Repeat until either k=1 or produce certificate that k-1 components is not realizable thus current realization has minimum number of components



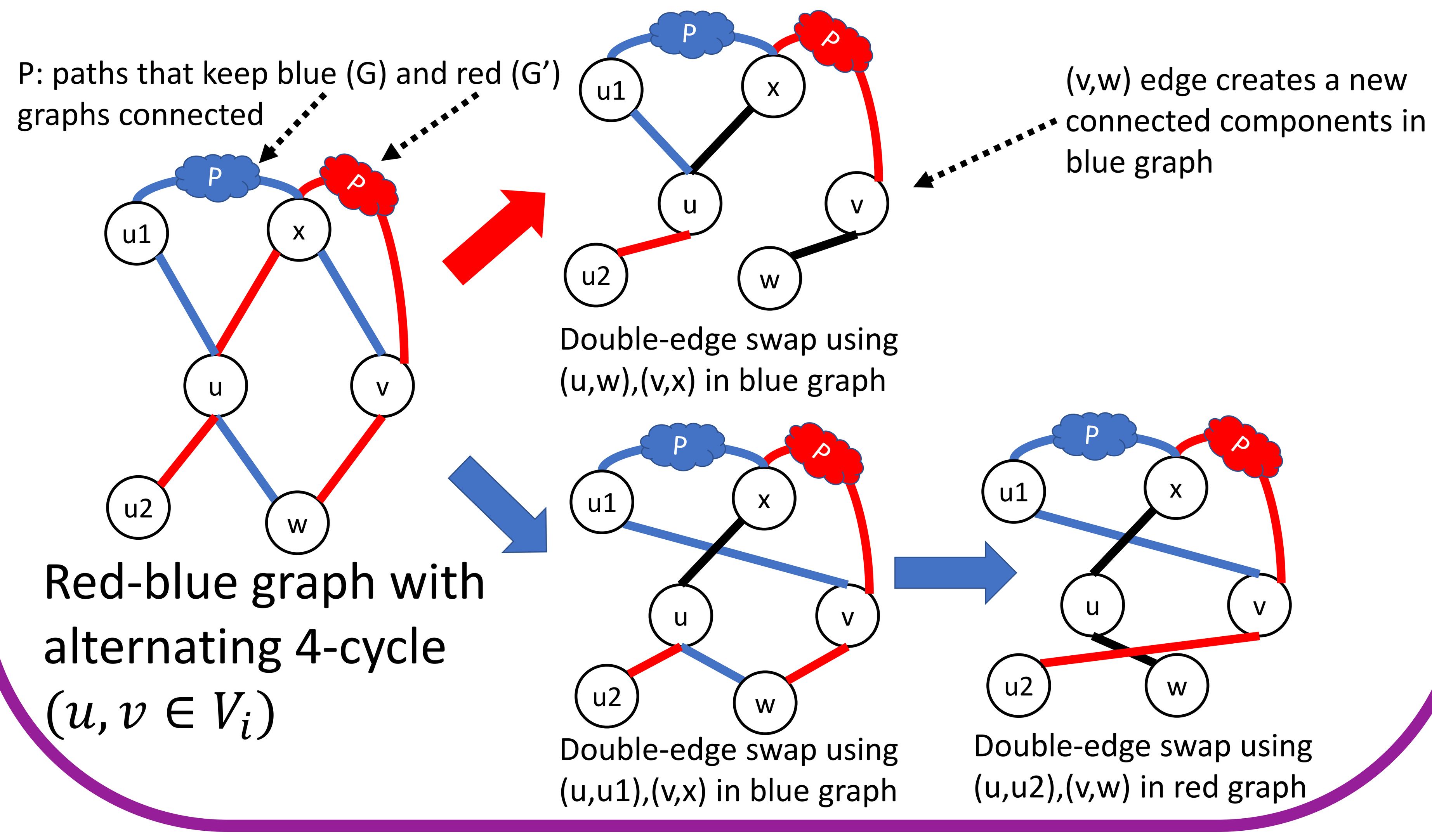
Key idea: JDM-preserving double-edge swap using (u,w),(v,x) breaks a cycle to merge two connected components ($u, v \in V_i$)

Proof sketch of Lemma 2

- Symmetric difference of G and G' (red-blue graph):
  - Exists a red-blue alternating cycle decomposition



- Goal: Given two realizations G, G', each with $k_{\min}$ components, find JDM-preserving double-edge swap(s) to reduce the symmetric difference without increasing number of connected components in G or G'
- *Example case:*



NetSci'18, Poster #440 – This work was supported by NSF IIS #1526736 – More at: athinagroup.eng.uci.edu